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LETTER TO THE EDITOR

Growing partially directed self-avoiding walks

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Abstract. A partially directed self-avoiding walk model with the 'kinetic growth' weighting is solved exactly, on the square lattice and for two restricted, strip geometries. Some finite-size effects are examined.

Recently various 'kinetically' growing walk models have been proposed (Amit *et al* 1983, Hemmer and Hemmer 1984, Majid *et al* 1984, Kremer and Lyklema 1984a,b, Weinrib and Trugman 1984) as part of a general study of dynamic aggregation (see Family and Landau 1984, for the literature). In this letter we consider the partially directed self-avoiding walk (SAW) on the square lattice, with allowed steps along $+\hat{x}$, $-\hat{x}$ and $+\hat{y}$. The usual 'configurational' versions of several directed SAW models are exactly solvable (Fisher and Sykes 1959, Blöte and Hilhorst 1983, Cardy 1983, Redner and Majid 1983, Szpilka 1983), and the partially directed SAW exhibit peculiar finite-size effects (Szpilka and Privman 1983). In the growing-walk version, each step of the walk is weighted with the probability factor which is the inverse of the number of all allowed steps (number of unvisited allowed sites) *before* the step under consideration was actually done. The total weight, W, of a given n-step walk is a product of n consecutive step probabilities. All walks begin at the origin, and for convenience we will allow for a zero-step walk with $W = 1$.

One important property of all the growth models is that the walk should never be trapped. The directed SAW never gets trapped *locally*. However, the above condition prevents defining a proper growth model for directed SAWs on *finite* lattices. We will consider finite-width strips which are infinite in one lattice direction. Another consequence of the non-trapping property (Nakanishi and Family 1984, see also Gould *et al* 1983) is that the 'susceptibility', $\chi(z)$, takes a simple form

$$\chi(z) \equiv \sum_{n=0}^{\infty} \sum_{|w|=n} W(w) z^n = 1/(1-z), \tag{1}$$

so that $z_c = 1$. Here the inner sum runs over all the allowed walks, w, of n steps ($|w| = n$). To every n-step walk the activity factor z^n is assigned in (1).

If we considered the *fully* directed walks, with only $+\hat{x}$ and $+\hat{y}$ steps allowed, then the difference between the configurational versus 'kinetic' description would be trivial, the latter being obtained by replacing $z^{(\text{conf.})} \to \frac{1}{2} z^{(\text{growth})}$ in all the quantities of interest. In particular, $z_c^{(\text{conf})} = \frac{1}{2}$ goes over to $z_c^{(\text{growth})} = 1$. The exponents describing the divergence of, say, $\xi_{\parallel}(z)$ and $\xi_{\perp}(z)$ as $z \to z_c^-$ remain unchanged ($\nu_{\parallel} = 1$ and $\nu_{\perp} = \frac{1}{2}$). By universality we then expect the bulk critical behaviour of the *partially* directed SAW model to remain unchanged by making it 'kinetic'. However, the behaviour away from z_c *is modified* non-trivially, see below (recall, $z_c^{(\mathrm{conf.})} = \sqrt{2}-1$), and even more dramatic changes are found in certain finite-size scaling properties.

As usual (see e.g. Szpilka 1983) we introduce the generating function

$$G(x_1, x_2, y) = \sum_{n=0}^{\infty} \sum_{|w|=n} W(w) x_1^{n_+} x_2^{n_-} y^{n-n_+-n_-}, \tag{2}$$

where n_+ and n_- denote the number of $+\hat{x}$ and $-\hat{x}$ steps, respectively, in the n-step walk. Then $G(z, z, z) \equiv \chi(z) = (1-z)^{-1}$ which provides a consistency check. The quantities which play the role of the correlation lengths for this problem can be defined via

$$\xi_\parallel(z) = \chi^{-1}(z) \sum_{n=0}^{\infty} \sum_{|w|=n} W(w) Y(w) z^n, \tag{3}$$

$$\xi_\perp^2(z) = \chi^{-1}(z) \sum_{n=0}^{\infty} \sum_{|w|=n} W(w) X^2(w) z^n, \tag{4}$$

where (X, Y) are the end-point coordinates, and for $\xi_\perp$ the quadratic moment is used because the linear one vanishes by symmetry.

Then

$$\xi_\parallel(z) = z[(\partial/\partial y) \ln G(x_1, x_2, y)]_{x_1, x_2, y \to z}, \tag{5}$$

$$\begin{aligned}\xi_\perp^2(z) = z^2\chi^{-1}(z)[(\partial/\partial x_1 - \partial/\partial x_2)^2 G(x_1, x_2, y)]_{x_1, x_2, y\to z} \\ + 2z[(\partial/\partial x_1) \ln G(x_1, x_2, y)]_{x_1, x_2, y\to z}.\end{aligned} \tag{6}$$

We will consider two finite-width strip geometries: (a) walk restricted to $0 \leq x \leq L_\perp$ (and $0 \leq y < \infty$) with *periodic* boundary conditions, namely the $+\hat{x}$ step from $(L_\perp, y)$ reaches the point $(0, y)$, while $(0, y) \to (L_\perp, y)$ is a $-\hat{x}$ step. (b) Walk restricted to $0 \leq y \leq L_\parallel$ (and $-\infty < x < \infty$) with *open* boundary at $y = L_\parallel$, so that the walk which reaches $(x, L_\parallel)$ can proceed only by $+\hat{x}$ steps or by $-\hat{x}$ steps.

Let us consider first the geometry (a). The infinite square lattice results will obtain as $L \to \infty$ (we write L in place of $L_\perp$, for simplicity). The generating function for walks which do not make $+\hat{y}$ steps is given by

$$G^{(x)} = 1 + \left(\frac{x_1}{3} + \frac{x_1^2}{3\cdot 2} + \frac{x_1^3}{3\cdot 2^2} + \ldots + \frac{x_1^L}{3\cdot 2^{L-1}}\right) + (x_1 \to x_2) \tag{7}$$

where the terms are self-explanatory: the probability factors for $(n>0)$-step walks are $\frac{1}{2}$ per step (two neighbours) except for the first step, when three allowed neighbours are unoccupied. The generating function for walks which make exactly one $+\hat{y}$ step: their *last* step, is given by

$$G^{(y)} = \frac{y}{3} + \left(\frac{x_1}{3}\cdot\frac{y}{2} + \frac{x_1^2}{3\cdot 2}\cdot\frac{y}{2} + \frac{x_1^3}{3\cdot 2^2}\cdot\frac{y}{2} + \ldots + \frac{x_1^L}{3\cdot 2^{L-1}}\cdot y\right) + (x_1 \to x_2), \tag{8}$$

where only the last, $\sim x_1^L$, term has a factor of y, in place of $y/2$. In terms of $G^{(x)}$ and $G^{(y)}$ we have

$$G = G^{(x)} + G^{(y)}G^{(x)} + G^{(y)^2}G^{(x)} + \ldots = G^{(x)}/(1 - G^{(y)}) \tag{9}$$

where

$$G^{(x)}=\frac{2}{3}\left(\frac{1-(x_1/2)^{L+1}}{1-(x_1/2)}+\frac{1-(x_2/2)^{L+1}}{1-(x_2/2)}-\frac{1}{2}\right) \tag{10}$$

$$G^{(y)}=\frac{y}{3}\left(\frac{1-(x_1/2)^{L+1}}{1-(x_1/2)}+\frac{1-(x_2/2)^{L+1}}{1-(x_2/2)}+\left(\frac{x_1}{2}\right)^L+\left(\frac{x_2}{2}\right)^L-1\right). \tag{11}$$

For the bulk system ($L\to\infty$) one then calculates (see (5) and (6))

$$\xi_\parallel(z;L=\infty)=\frac{z(2+z)}{(1-z)(6+z)}\approx\frac{\text{constant}}{1-z},\qquad \text{as } z\to z_c^-, \tag{12}$$

$$\xi_\perp^2(z;L=\infty)=\frac{2z(z^2+2z+4)}{(1-z)(6+z)(2-z)}\approx\frac{\text{constant}}{1-z},\qquad \text{as } z\to z_c^-, \tag{13}$$

where $z_c=1$. Thus $\nu_\parallel=1$ and $2\nu_\perp=1$ as anticipated. For finite $L\equiv L_\perp$,

$$\xi_\parallel(z;L_\perp)=\frac{z[2+z+4(z/2)^L-8(z/2)^{L+1}]}{(1-z)[6+z-8(z/2)^{L+1}]}. \tag{14}$$

We found that the closed-form calculation of $\xi_\perp^2(z;L<\infty)$ is intractably complicated. Therefore, we will examine the finite-size behaviour of $\xi_\parallel(z;L)$ only, both here and for the geometry (b). We observe that $\xi_\parallel(z;L_\perp)$ diverges at the bulk critical point, at $z=1$. One finds

$$\xi_\parallel(z;L_\perp)/\xi_\parallel(z;\infty)=1+O(z_c-z)+O(2^{-L_\perp}), \tag{15}$$

so that the finite-size effects enter only in corrections to scaling! Indeed, the standard finite-size scaling hypothesis for $\xi_\parallel$, in this geometry, reads

$$\xi_\parallel(z;L_\perp)/\xi_\parallel(z;\infty)\approx Y^{(A)}(L_\perp/\xi_\perp(z;\infty)),\qquad \text{as } z\to z_c^-, \tag{16}$$

see Fisher (1971), Fisher and Barber (1972), and a review by Barber (1983). For the configurational version of this problem, one finds instead of $L_\perp/\xi_\perp(z;\infty)$, a different, anomalous scaling combination in which $L_\perp$ enters through a new, exponential (in $L_\perp$) *longitudinal* length scale which 'scales' with $\xi_\parallel(z;\infty)$, see Szpilka and Privman (1983), for details. In the 'kinetic' case, relation (15) suggests $Y^{(A)}\equiv 1$, so that the scaling argument remains undetermined. More generally, however, one may expect no anomalous length scale entering because it is normally related to the value of $\xi_\parallel(z;L_\perp)$ *at* the bulk z_c (Privman and Fisher 1983). This value, $\xi_\parallel(z_c;L_\perp)$, is *infinite* in the 'kinetic' model.

Consider next the geometry (b) where the walk is allowed to make no more than $L_\parallel$ steps along $+\hat{y}$. Relation (9) is replaced by

$$\begin{aligned} G&=(G^{(x)}+G^{(y)}G^{(x)}+G^{(y)^2}G^{(x)}+\ldots+G^{(y)^{L_\parallel-1}}G^{(x)})+G^{(y)^{L_\parallel}}\bar{G}^{(x)}\\ &=G^{(x)}(1-G^{(y)^{L_\parallel}})/(1-G^{(y)})+\bar{G}^{(x)}G^{(y)^{L_\parallel}}, \end{aligned} \tag{17}$$

where $G^{(x)}$ and $G^{(y)}$ are with $L\to\infty$ in (10)-(11), while $\bar{G}^{(x)}$ is the generating function for walking *at* $y=L_\parallel$, given by

$$\begin{aligned} \bar{G}^{(x)}&=1+\left(\frac{x_1}{2}+\frac{x_1}{2}x_1+\frac{x_1}{2}x_1^2+\ldots\right)+(x_1\to x_2)\\ &=\frac{1}{2}\left(\frac{1}{1-x_1}+\frac{1}{1-x_2}\right). \end{aligned} \tag{18}$$

For $\xi_\parallel(z; L_\parallel)$ we obtain, by (5),

$$\xi_\parallel(z; L_\parallel) = \xi_\parallel(z; \infty)\left[1 - \left(\frac{z(2+z)}{3(2-z)}\right)^{L_\parallel - 1}\right], \tag{19}$$

where $\xi_\parallel(z; \infty)$ is given by (12). As $z \to z_c^-$ and $L_\parallel \to \infty$, we anticipate, in place of (16),

$$\xi_\parallel(z; L_\parallel)/\xi_\parallel(z; \infty) \approx Y^{(B)}(L_\parallel/\xi_\parallel(z; \infty)). \tag{20}$$

A straightforward analysis of (19) verifies this relation with

$$Y^{(B)}(\tau) = 1 - e^{-\tau}. \tag{21}$$

At the bulk z_c,

$$\xi_\parallel(z_c; L) \approx L \tag{22}$$

so that asymptotically $\xi_\parallel$ attains its largest possible value for this geometry.

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